

On Haar systems for groupoids

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Abstract: It is shown that a locally compact groupoid with open range map does not always admit a Haar system. It then is shown how to construct a Haar system if the stability groupoid and the quotient by the stability groupoid both admit one.

MSC: 28C10, 22A22

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Introduction

Topological groupoids occur naturally in encoding hidden symmetries like in fundamental groupoids or holonomy groupoids of foliations, see [Pat99], for instance. In order to construct convolution algebras on groupoids [Ren80, KPRR97], one needs continuous families of invariant measures, so called *Haar systems* [Sed76], see also Section 2. These do not always exist. One known criterion is that a Haar system can only exist if the range map is open (Corollary to Lemma 2 in [Sed86], see also [Wil15]).

A second criterion, which has been neglected in the literature, is the possibility of *failing support*, i.e., it is possible that, although the range map is open, the support condition of a Haar system cannot be satisfied, see Proposition 2.10. It is believed, however, that there should always be a Haar system for a locally compact groupoid with open range map, if the groupoid is second countable.

We show how to construct Haar systems if the stability groupoid and its quotient both admit one.

I thank Dana Williams for his very helpful comments.

1 A relative Urysohn Lemma

For the convenience of the reader, and to settle the notation, we include a folklore lemma here. Let $X = \bigcup_{i \in I} U_i$ be an open covering of a topological space X . Recall a *refinement* of the covering $(U_i)_{i \in I}$ is another covering $(V_j)_{j \in J}$ such that for each $j \in J$ there exists $i \in I$ with $V_j \subset U_i$.

Recall that a topological space is called *paracompact*, if every open cover admits a locally finite refinement. It is known that compact spaces are paracompact, second countable locally compact Hausdorff spaces are paracompact [Mor61], CW-complexes are paracompact and metric spaces are paracompact [Rud69].

Lemma 1.1. *Let $\pi : Y \rightarrow X$ be a continuous and open map between locally compact Hausdorff spaces and let $\sigma : X \rightarrow Y$ be a continuous section, i.e., one has $\pi \circ \sigma = \text{Id}_X$. Let U, V be two open neighborhoods of $\sigma(X)$ such that $\overline{U} \subset V$ and $\pi^{-1}(x) \cap \overline{U}$ is compact for every $x \in X$.*

If X is paracompact, then there exists a continuous map $f : Y \rightarrow [0, 1]$ such that $f \equiv 1$ in U and $\pi^{-1}(x) \cap \text{supp } f$ is a compact subset of V for every $x \in X$.

Proof. Let Y^+ denote the disjoint union

$$Y^+ = X \cup Y$$

and equip Y^+ with the following topology: A given set $W \subset Y^+$ is open if and only if

- $W \cap X$ is open in X ,
- $W \cap Y$ is open in Y and
- for every compact $K \subset X \cap W$ the set $\pi^{-1}(K) \setminus W$ is compact in Y .

This defines a topology on Y^+ . We claim that Y^+ is a locally compact Hausdorff space. The Hausdorff property is easy, so we concentrate on showing that each point has a compact neighborhood. First let $y \in Y$, then any neighborhood $K \subset Y$ of y , which is compact in Y , remains compact in Y^+ . So consider $x \in X$ and let $W \subset X$ be an open neighborhood of x with compact X -closure $\overline{W}^X \subset X$. The set $\sigma(\overline{W}^X)$ is compact in Y and there exists an open neighborhood $O \subset Y$ of $\sigma(\overline{W}^X)$ with compact Y -closure \overline{O}^Y . Let

$$R = W \cup \left(\pi^{-1}(W) \setminus \overline{O}^Y \right).$$

It is clear that R is an open neighborhood of x in Y^+ . We claim that it has compact closure $\overline{R} \subset Y^+$. Observe that the set R is contained in

$$A = \overline{W}^X \cup \left(\pi^{-1}(\overline{W}^X) \setminus O \right),$$

which is easily seen to be compact in Y^+ .

Next we make use of the fact that X is paracompact. Let $(U_i)_{i \in I}$ be an open covering of X by relatively compact sets and let $(u_i)_{i \in I}$ be an underlying partition of unity. For each i the set $Y_i = \pi^{-1}(\overline{U_i}^X) \cup \overline{U_i}^X$ is compact in Y^+ . By Urysohn's Lemma there exists a continuous function $f_i : Y_i \rightarrow [0, 1]$ such

that $f_i \equiv 1$ in U and f_i vanishes outside V . The function $u_i f_i$ extends by zero to a continuous function on Y^+ and the function

$$f = \sum_{i \in I} u_i f_i$$

satisfies the lemma. \square

2 Locally compact groupoids and Haar systems

Definition 2.1. By a *group bundle* we understand a continuous map $\pi : G \rightarrow X$ between locally compact Hausdorff spaces together with a group structure on each fibre $G_x = \pi^{-1}(x)$, $x \in X$ such that the following maps are continuous:

$$\begin{array}{ll} \varepsilon : X \rightarrow G & \text{identity,} \\ m : G^{(2)} \rightarrow G & \text{multiplication,} \\ \iota : G \rightarrow G & \text{inverse,} \end{array}$$

where $G^{(2)}$ is the set of all $(x, y) \in G \times G$ with $\pi(x) = \pi(y)$.

Note that this implies that ε is a homeomorphism onto the image, so X carries the subspace topology but also X carries the quotient topology induced by the surjective map π . In all, the topology on X is determined by the one on G .

Note that a group bundle is a special case of a locally compact groupoid, we shall get back to this in Section 2.

Definition 2.2. Each fibre G_x being a locally compact group, carries a Haar measure which is unique up to scaling. A *coherent system* of Haar measures is a family $(\mu_x)_{x \in X}$ such that μ_x is a Haar measure on G_x such that for each $\phi \in C_c(G)$ the map

$$x \mapsto \int_{G_x} \phi d\mu_x$$

is continuous.

Proposition 2.3. *Let $\pi : G \rightarrow X$ be a group bundle over a paracompact space X . There exists a coherent system of Haar measures μ_x on G if and only if the map π is open.*

Proof. This is Lemma 1.3 in [Ren91]. □

Definition 2.4. Let X be a set. By a *groupoid* over X we mean a category with object class X (so it is a small category) in which each arrow is an isomorphism. We write G for the set of arrows and we use the following notation

$r, s : G \rightarrow X$	range and source maps,
$\varepsilon : X \rightarrow G$	identity,
$G^{(2)} \subset G \times G$	set of composable pairs,
$m : G^{(2)} \rightarrow G$	composition,
$\iota : G \rightarrow G$	inverse.

Definition 2.5. A *topological groupoid* is a groupoid G over X together with topologies on G and X such that the structure maps $r, s, \varepsilon, m, \iota$ are continuous. Here $G \times G$ carries the product topology and $G^{(2)} \subset G \times G$ the subset topology. Note that if X is Hausdorff, then $G^{(2)} = \{(\alpha, \beta) \in G \times G : r(\beta) = s(\alpha)\}$ is a closed subset of $G \times G$.

A *locally compact groupoid* is a topological groupoid such that G and X are locally compact Hausdorff spaces.

From now on G is assumed to be a locally compact groupoid. We use the notation

$$\begin{aligned} G_x &= \{g \in G : s(g) = x\}, \\ G^y &= \{g \in G : r(g) = y\}, \\ G_x^y &= G_x \cap G^y. \end{aligned}$$

As X is Hausdorff, all three sets are closed in G .

Note that a group bundle is a special case of a groupoid G with $G_x^y = \emptyset$ if $x \neq y$.

Definition 2.6. For a groupoid G the *stability groupoid* is defined to be the subset

$$G' = \{g \in G : r(g) = s(g)\}.$$

If G is a topological groupoid, then G' is a closed subgroupoid.

Definition 2.7. On a groupoid G we instal an equivalence relation

$$g \sim h \iff r(g) = r(h) \text{ and } s(g) = s(h).$$

we write $[g]$ for the equivalence class, i.e., $[g] = G_{s(g)}^{r(g)}$.

Now assume that $(\mu_x^x)_{x \in X}$ is a coherent family of measures on the group bundle $G' = \{g \in G : r(g) = s(g)\}$. We then get invariant measures $\mu_{[g]}$ on the classes $[g]$ by setting

$$\int_{[g]} \phi(x) d\mu_{[g]}(x) = \int_{G_{s(g)}^{s(g)}} \phi(gx) d\mu_{s(g)}^{s(g)}(x).$$

The invariance of the μ_x^x yields the well-definedness of the $\mu_{[g]}$. The uniqueness of a Haar measure implies that $\mu_{[g]}$ is, up to scaling, the unique Radon measure on $[g]$ being right-invariant under $G_{s(g)}^{s(g)}$ or left-invariant under $G_{r(g)}^{r(g)}$.

In the sequel, we shall identify a Radon measure with its positive linear functional, so we write $\mu_{[g]}(\phi)$ for the above integral.

Lemma 2.8. *Let G be a locally compact groupoid over a paracompact space X and let $(\mu_x^x)_{x \in X}$ be a coherent system of Haar measures on the groups G_x^x , $x \in X$. Then for every $\phi \in C_c(G)$ the function*

$$\overline{\phi} : g \mapsto \mu_{[g]}(\phi)$$

is continuous.

Proof. The proof is quite similar to the proof of Lemma 1.3 in [Ren91]. For the convenience of the reader we give it nevertheless. The existence of a coherent system implies that the range map on the group bundle G' is open. Therefore, by Lemma 1.1 there exists a continuous map $f_0 : G \rightarrow [0, 1]$ such that $f_0 \equiv 1$ in a neighborhood of the diagonal $\Delta = \varepsilon(X)$ and $G^x \cap \text{supp}(f)$ is compact for every $x \in X$. We can normalize the system (μ_x^x) by requiring

$\mu_x^x(f_0) = 1$ for every $x \in X$. For every $\phi \in C_c(G)$ let $(\phi : f_0)$ denote the infimum of all numbers $\sum_{j=1}^n \lambda_j$ where $\lambda_j > 0$ is such that there are $g_j \in G$ with

$$|\phi(g)| \leq \sum_{j=1}^n \lambda_j f_0(g_j g)$$

holds for all $g \in G$. It then follows $\mu_{[g]}(\phi) \leq (\phi : f_0)$ for every $g \in G$. So the map $g \mapsto \mu_{[g]}(\phi)$ is bounded for every given $\phi \in C_c(G)$. Now let $(g_i)_{i \in I}$ be a net in G converging to $g \in G$. Let $\mathcal{C} \subset \ell^\infty(I)$ be the subset of convergent nets in \mathbb{C} with index set I . The functional $\lim_{i \in I}$ which assigns the limit to a convergent net, is positive and continuous with respect to the ℓ^∞ topology, hence can be extended to a positive continuous linear functional $\omega : \ell^\infty(I) \rightarrow \mathbb{C}$. Let $\phi \in C_c(G)$ be given, then $\mu_{[g_i]}(\phi)$ lies in $\ell^\infty(I)$. We claim that $\omega(\mu_{[g_i]}(\phi))$ equals $\mu_{[g]}(\phi)$, independent of ω , which proves the convergence $\mu_{[g_i]}(\phi) \rightarrow \mu_{[g]}(\phi)$. First observe that $\phi \mapsto \omega(\mu_{[g_i]}(\phi))$ is a positive linear functional, hence a Radon measure. Next we show that it only depends on $\phi|_{[g]}$, so it has support in the closed set $[g]$. Let $\psi : G \rightarrow [0, 1]$ be continuous of compact support with $\psi \equiv 1$ on the compact set $([g] \cup \bigcup_{i \in I} [g_i]) \cap \text{supp}(\phi)$. Next suppose $\phi|_{[g]} \equiv 0$ and let $\varepsilon > 0$. Then there exists a compact neighborhood U of $\text{supp}(\phi) \cap [g]$ such that $|\phi| < \varepsilon$ on U . By compactness, there exists $i_0 \in I$ such that for all $i \geq i_0$ one has $([g_i] \cap \text{supp}(\phi)) \subset U$. Hence for $i \geq i_0$ we have $|\mu_{[g_i]}(\phi)| < \varepsilon \mu_{[g_i]}(\psi) \leq \varepsilon C$, where $C = \sup_{h \in G} \mu_{[h]}(\psi)$. So we have that the Radon measure $\omega(\mu_{[g_i]}(\phi))$ is supported in $[g]$. We show that it is G_x^x -invariant, where $x = r(g)$. For this let $\alpha \in G_x^x$. The range map for G' is open by Proposition 2.3, hence there exist $\alpha_i \in G_{r(g_i)}^{r(g_i)}$ such that α_i converges to α . The invariance of $\mu_{[g_i]}$ under α_i then implies the invariance of $\mu_{[g]}$ under α . By uniqueness, $\omega(\mu_{[g_i]})$ must be a scalar multiple of $\mu_{[g]}$. As both have value 1 on f_0 , the scalar equals 1. \square

Definition 2.9. A *Haar system* on the locally compact groupoid G is a family $(\mu^x)_{x \in X}$ of Radon measures on G with

- (a) $\text{supp}(\mu^x) = G^x$,
- (b) $\int_G \phi(\alpha g) d\mu^y(g) = \int_G \phi(g) d\mu^x$ for every $\phi \in C_c(G)$ and every $\alpha \in G_y^x$,
- (c) $x \mapsto \int_G \phi(g) d\mu^x(g)$ is continuous on X for every $\phi \in C_c(G)$.

If a locally compact groupoid G admits a Haar system, then the range map, and so the source map, too, is open, see Corollary to Lemma 2 in [Sed86], see also [Wil15].

The question for the converse assertion, asked in [Wil15], is answered in the negative by the following proposition.

Proposition 2.10. *There exists a locally compact, even compact, groupoid G , whose range map is open, but no Haar system exists on G .*

Proof. There are locally compact, even compact, Hausdorff spaces which cannot be the support of any Radon measure. Here are two examples:

- Let X be the unit ball of a Hilbert space of uncountable dimension and equip X with the weak topology. By the Banach-Alaoglu-Theorem, X is a compact Hausdorff space. By Corollary 7.14.59 of volume 2 of [Bog07], the set X cannot be the support of any Radon measure
- (Williams) Let Y be an uncountable set with the discrete topology and let $X = Y \cup \{\infty\}$ be its one-point compactification. Then X cannot be the support of any Radon measure. To see this, let m be a Radon measure on X , then $m(X) < \infty$, as X is compact. Further, $m(Y) = \sum_{y \in Y} m(\{y\})$, as m is regular and the only compact subsets of Y are the finite sets. As $m(Y) < \infty$, the set M of all $y \in Y$ with $m(\{y\}) > 0$ is countable, therefore $M \neq Y$ and m is supported in $M \cup \{\infty\}$.

Let now X be any locally compact Hausdorff space which is not the support of a Radon measure. Let $G = X \times X$ with the product topology and make g a groupoid by setting $(x, y)(y, z) = (x, z)$ and $r(x, y) = x$ as well as $s(x, y) = y$. Then the source map is a homeomorphism between G^x and X , so G^x cannot be the support of any Radon measure, hence no Haar system exists. \square

Definition 2.11. Let G be a groupoid over X . We write $E(G) \subset X \times X$ for the image of the map $g \mapsto (s(g), r(g))$. Then $E(G)$ is an equivalence relation on X .

We say that a groupoid G is an *equivalence groupoid* if $G_x^x = \{1_x\}$ for every $x \in X$. This means that the groupoid is completely described by its equivalence relation. Note, though, that for topological groupoids the topology on G generally differs from the one on $E(G)$ as a subset of $X \times X$.

Lemma 2.12. *Let G be a groupoid over a set X . Define an equivalence relation on G by*

$$g \sim h \iff r(g) = r(h) \text{ and } s(g) = s(h).$$

Then the set $\overline{G} = G / \sim$ becomes a groupoid by setting $[g][h] = [gh]$ whenever g and h are composable.

Proof. This is easily checked. \square

Theorem 2.13. *Let G be a locally compact groupoid over a paracompact space X . Suppose that the stability groupoid G' has open range map.*

- (a) *The groupoid \overline{G} , when equipped with the quotient topology, is a locally compact groupoid. The quotient map $G \rightarrow \overline{G}$ is open.*
- (b) *If the range map of G is open, then so is the range map of \overline{G} .*
- (c) *If \overline{G} admits a Haar system, then G admits a Haar system.*
- (d) *If G is second countable and the equivalence $E(G)$ is a closed subset of $X \times X$, then G admits a Haar system.*

Proof. (a) By Proposition 2.3, the groupoid G admits a coherent system of Haar measures $(\mu_x^x)_{x \in X}$. Let $g_0 \in G$ and let $\phi \in C_c^+(G)$ such that $\phi(g_0) > 0$. Let

$$\overline{\phi} : g \mapsto \int_{s(g)}^{r(g)} \phi(gh) d\mu_{s(g)}^{r(g)}(h).$$

By Proposition 2.8 the map $\overline{\phi}$ is continuous. It factors over \overline{G} , hence defines a continuous map of compact support on \overline{G} . The set $U = \{x \in \overline{G} : \overline{\phi}(x) > 0\}$ is an open neighborhood of $[g_0]$, so $\text{supp}(\overline{\phi})$ is a compact neighborhood of $[g_0]$. Therefore \overline{G} is locally compact.

If $[g] \neq [h]$, then we can find $\phi, \psi \in C_c^+(G)$ such that $\overline{\phi}$ and $\overline{\psi}$ have disjoint supports and $\phi(g), \psi(h) > 0$. Considering the continuous function $\overline{\phi} - \overline{\psi}$ on \overline{G} , one sees that $[h]$ and $[g]$ have disjoint neighborhoods, so \overline{G} is a Hausdorff space. Together we infer that \overline{G} is a locally compact groupoid.

Next we show that the quotient map $p : G \rightarrow \overline{G}$ is open. By Urysohn's lemma the topology on G has a basis consisting of the sets $U(\phi)$, where $\phi \in C_c^+(G)$

and $U(\phi = \{x \in G : \phi(x) > 0\})$. Then $p(U(\phi)) = U(\bar{\phi})$ is open again, so p is open.

(b) As the range map of G is open and factors over the range map of \bar{G} , the range map of \bar{G} is open as well.

(c) If (m^x) is a Haar system for \bar{G} , then

$$\phi \mapsto \int_{\bar{G}} \bar{\phi}(g) dm^x(g)$$

defines a Haar system on G .

(d) Using (c) and replacing G with \bar{G} , it suffices to show (d) under the condition that $G_x^x = \{1\}$ for every $x \in X$.

Assume this. On the set X we now define an equivalence relation

$$x \sim y \iff G_y^x \neq \emptyset.$$

Let $\bar{X} = X / \sim$. We claim that \bar{X} is a Hausdorff space. As we assume that $X^{(2)}$ is closed, we get that each equivalence class $[x] \subset X$ is closed and if $x_j \rightarrow x$ and $y_j \rightarrow y$ are convergent nets in X with $x_j \sim y_j$ for each j , then $x \sim y$. With these properties, the proof of the Hausdorff property of \bar{X} is the same as the proof of the Hausdorff property of \bar{G} above. Let $\pi : X \rightarrow \bar{X}$ be the projection map. By Lemma 2.3 of [Wil15] we conclude that there exists a family of Radon measures $(m_{[x]})$ on X , parametrized by \bar{X} such that $\text{supp}(m_{[x]}) = \pi^{-1}([x]) = [x]$ holds for every $x \in X$ and the function $[x] \mapsto \int_X \phi dm_{[x]}$ is continuous on \bar{X} . As G is a second countable locally compact Hausdorff space, hence a polish space, and \bar{X} is Hausdorff, it follows from Theorem 2.1 of [Ram90], that the source map induces a homeomorphism $G^x \xrightarrow{\cong} [x]$. So let $m^x = s_*^{-1}(m_{[x]})$. Then m^x is a Radon measure on G with support equal to G^x . For any $\alpha \in G_y^x$ the diagram

$$\begin{array}{ccc} G^y & \xrightarrow{s} & [y] = [x] \\ & \searrow \alpha & \nearrow s \\ & G^x & \end{array}$$

commutes, therefore the family m^x is indeed a Haar system. This concludes the proof of part (d) of the theorem. \square

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June 20, 2016